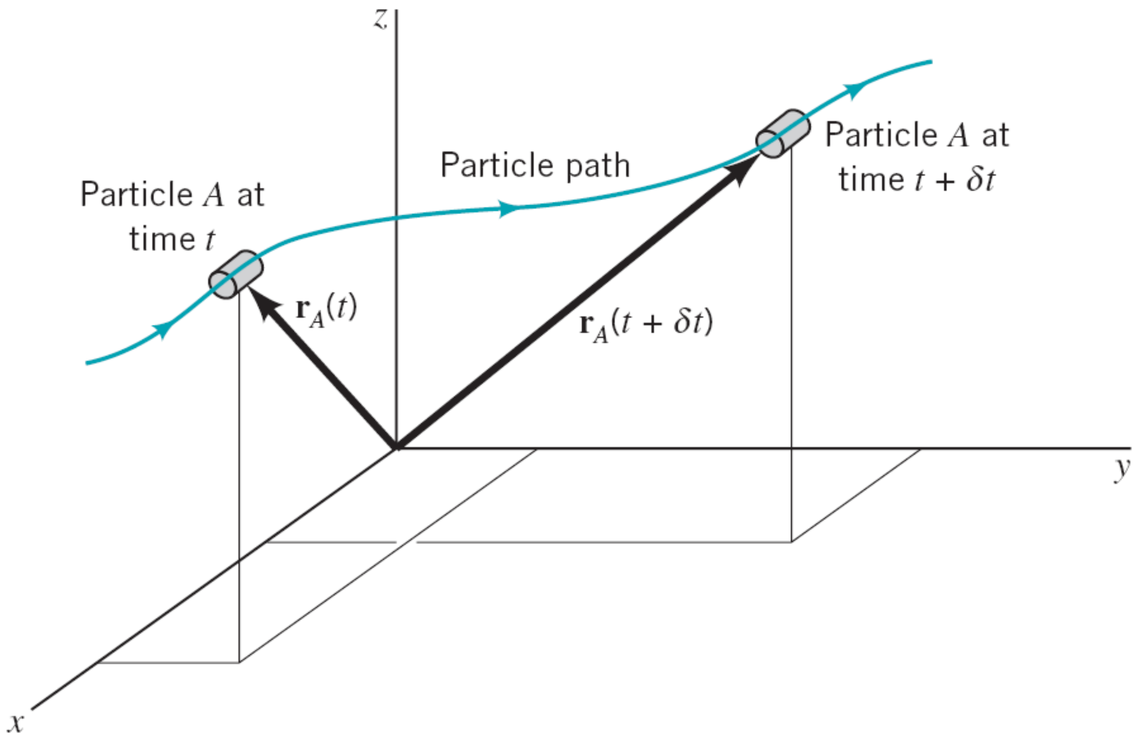
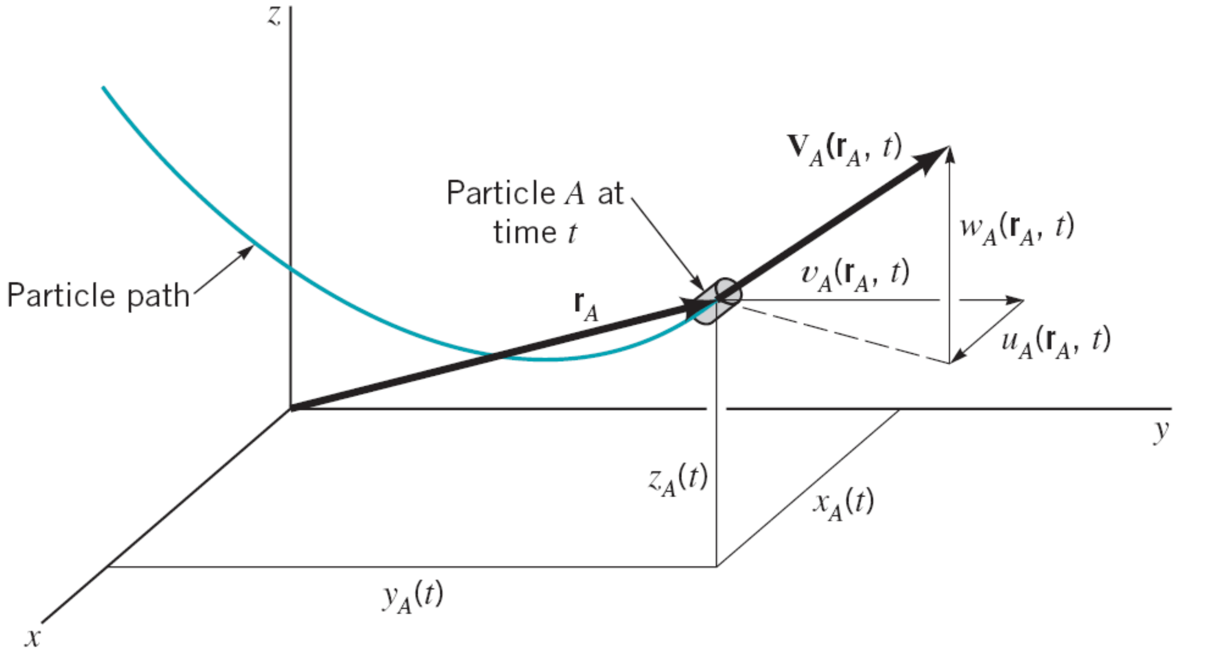
# The velocity field




$$\vec{V} = \lim_{\delta t \to 0} \frac{\vec{r}(t+\delta t) - \vec{r}(t)}{\delta t}$$

# The velocity vector



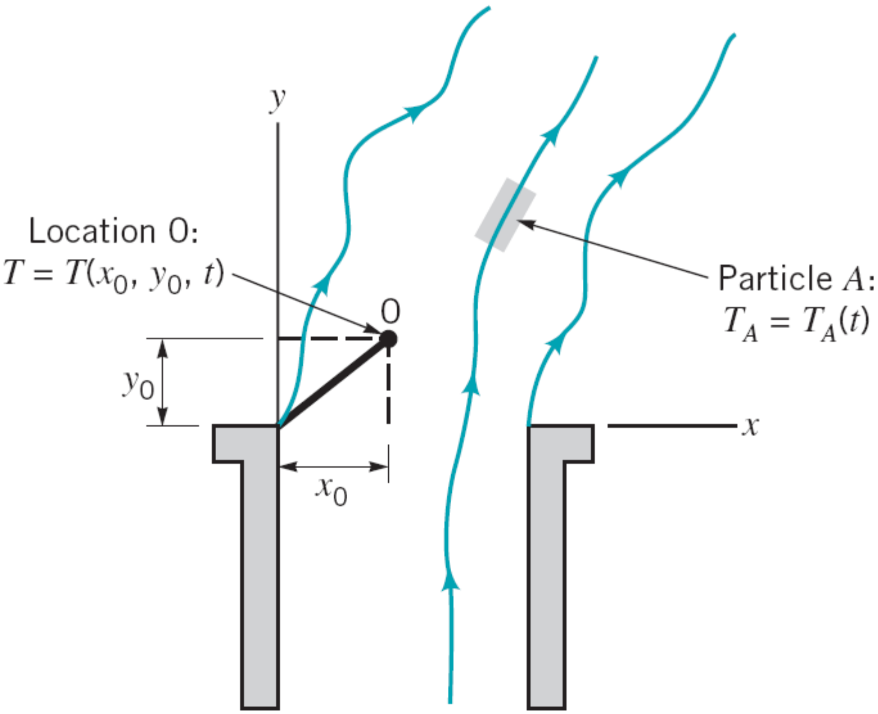

Velocity is a vector

$$\vec{V} = u(x,y,z,t)\,\hat{i} + v(x,y,z,t)\,\hat{j} + w(x,y,z,t)\,\hat{k}$$

$\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors in x, y and z directions

Speed: $|V| = \sqrt{u^2+v^2+w^2}$ scalar

# Eulerian and Lagrangian descriptions of a flowing fluid.




**Langrangian description of flow:**

Mark a particle and follow it in time as it flows in the flow field

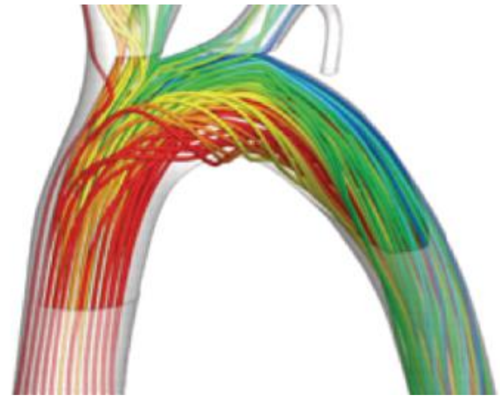
**Eulerian description of flow:**

Prescribe all flow quantities ($\vec{V}$, $p$, $\tau$, etc.) as functions of time ($t$) and space ($x, y, z$)

The Eulerian approach is the preferred method in fluid mechanics

# Flow field characterization

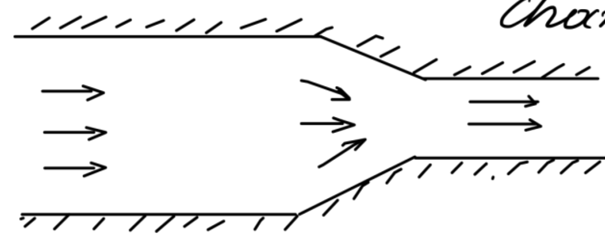
Flow field is → 1-D
→ 2-D
→ 3-D



3D flow in aorta

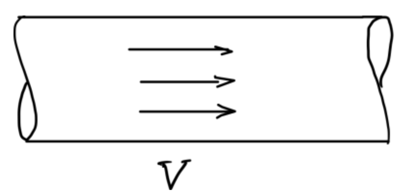
In general, the flow field is 3-D

Sometimes <u>one</u> (2-D)



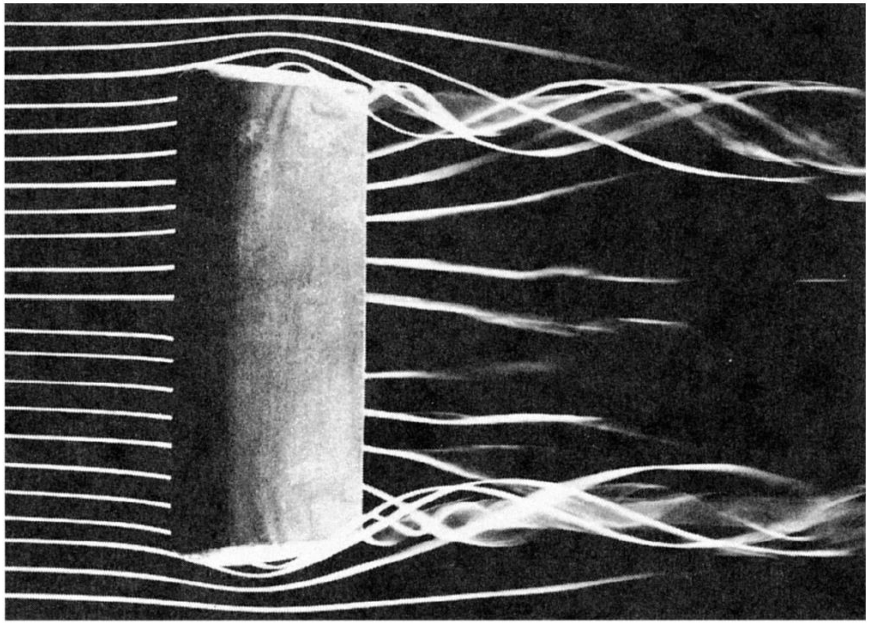
channel flow

or <u>two</u> velocity components (1-D)
are small




pipe flow

# Flow field examples



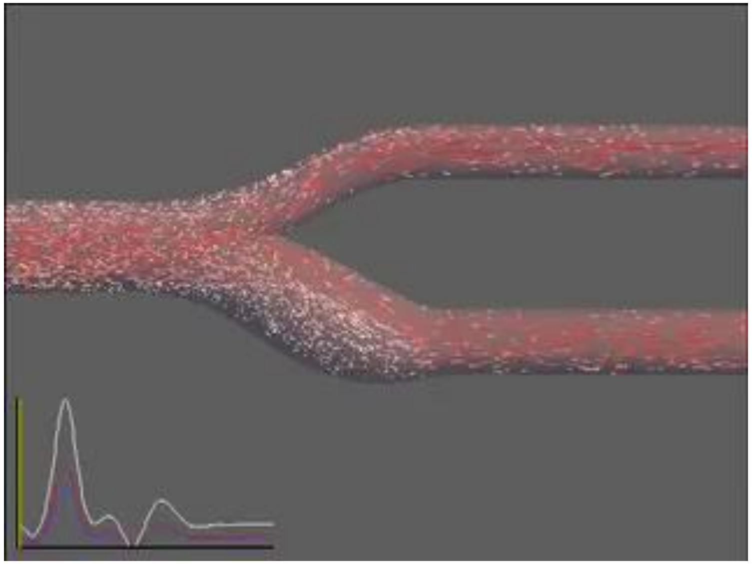
1D upstream, 3D downstream

# Steady vs. unsteady & laminar vs. turbulent flow

Steady flow:

Fluid properties at a given point do not change with time

i.e. $\frac{\partial V}{\partial t} = 0$ , $\frac{\partial P}{\partial t} = 0$ , $\frac{\partial \rho}{\partial t} = 0$



Unsteady flow:

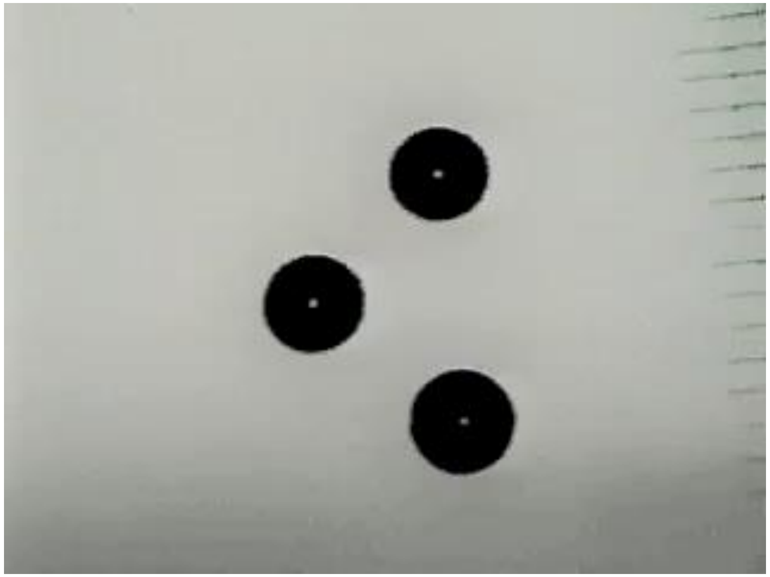
→ periodical (i.e. blood flow)

→ non-periodical

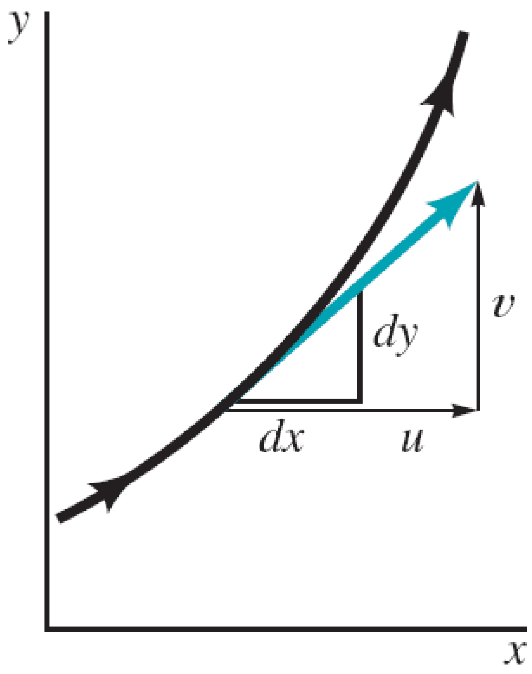
→ turbulent (random fluctuations)

Opposite: laminar

# Streamlines



Flow visualisation

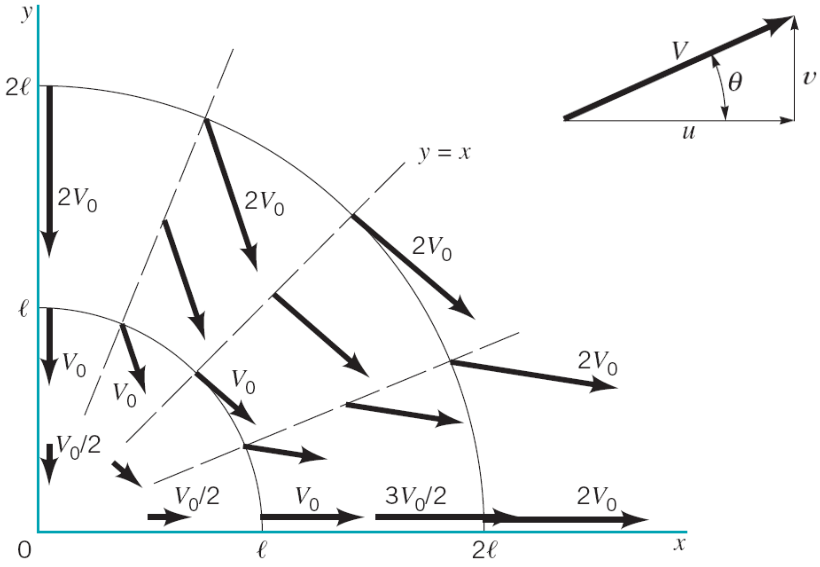


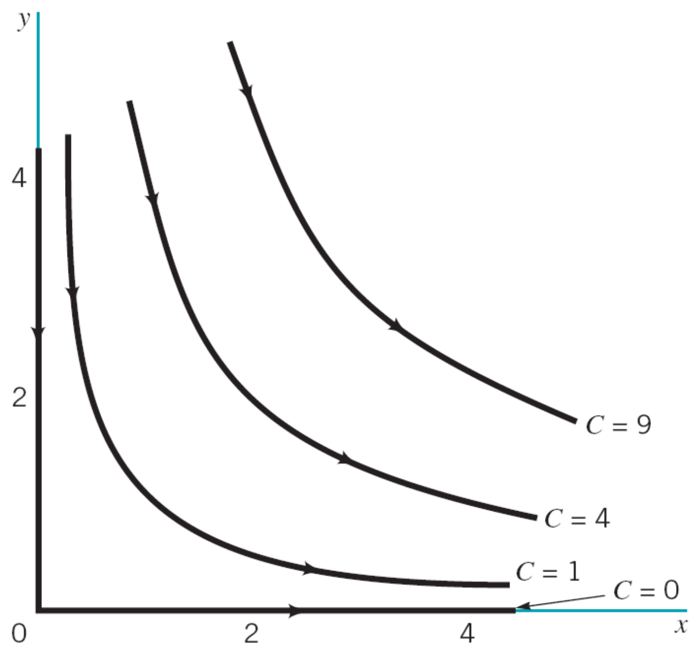

Streamlines: lines tangent to the velocity field

$$\frac{v}{u} = \frac{dy}{dx}$$

Solve this differential equation to get streamlines

# Example: 2-D flow field




$$u = V_0 x \qquad v = -V_0 y$$

$$\frac{dy}{dx} = \frac{v}{u} = -\frac{V_0 y}{V_0 x} = -\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x} \Rightarrow \frac{dx}{x} = -\frac{dy}{y}$$

$$\Rightarrow \ln x = -\ln y + c$$

$$\Rightarrow \ln x \cdot y = c$$

$$\Rightarrow x \cdot y = c'$$

# Streamlines, streaklines and pathlines

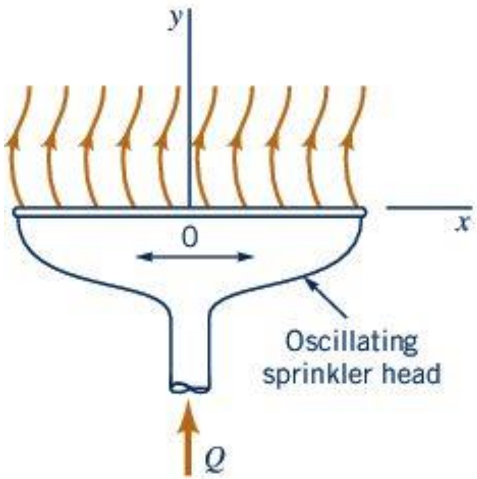
**Streamline:** Line that, at a given moment t, is everywhere tangential to the velocity field. Strictly Eulerian concept.

**Streakline:** Line formed by all particles that, at a given moment t, have previously passed through a common point. Most often used in a laboratory setting to visualize flow by injecting buoyant smoke in air or dye in water.

**Pathline:** Trajectory followed by a single particle that flows from one point to another. Strictly Lagrangian concept.

Note: In steady flow, all particles follow the same trajectory and thus each streakline coincides with a streamline through the injection point. Similarly, each pathline is de facto also a streakline since all subsequent particles will follow the same path. In unsteady flow, on the other hand, particles injected at $t=t_2$ do not necessarily follow the same trajectory as particles injected at $t=t_1$. Hence pathlines, streaklines and streamlines do not necessarily coincide.

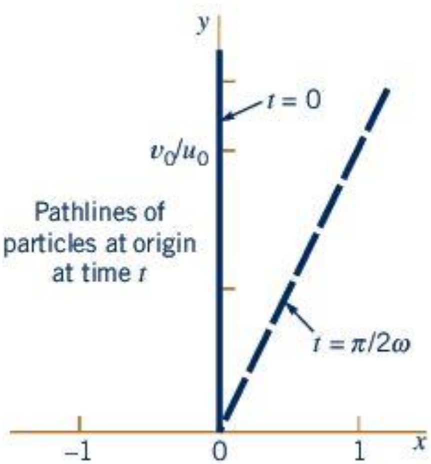
# Example



(a)

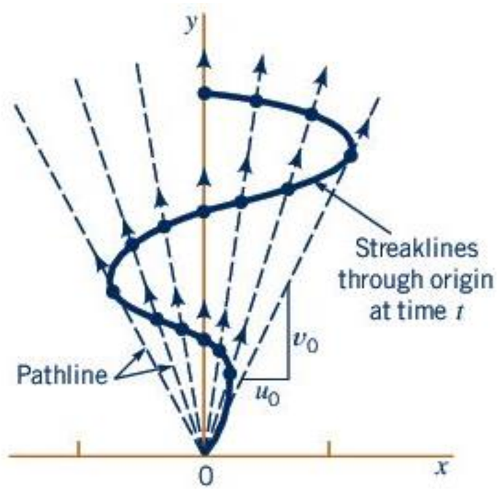
Q: Water particles flowing from a sinusoidally oscillating sprinkler head are ejected in straight rays. Discuss the difference between streamlines, pathlines and streaklines that pass through the origin.

Pathline



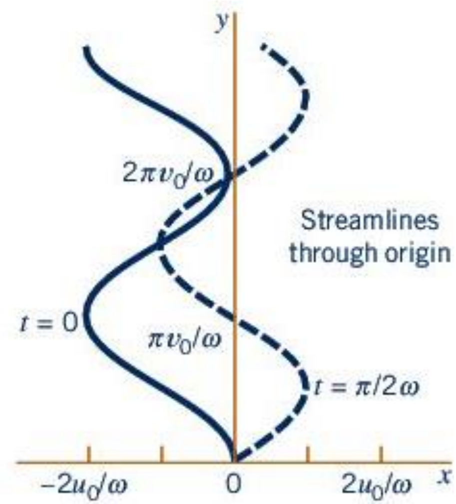
(c)

Streakline



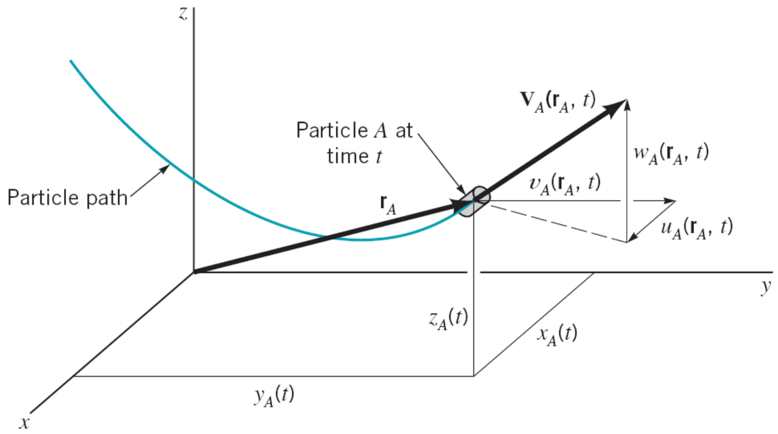
(d)

Streamline



(b)

# The acceleration field




<u>Eulerian approach</u>: describe the acceleration field as a function of position, $\vec{x}$ and time, $t$

<u>Material derivative</u>

The velocity of particle A, $V_A$ is:

$$\vec{V}_A = \vec{V}_A(\vec{r}_A, t)$$
$$= u_A(\vec{r}_A, t)\,\vec{i} + v_A(\vec{r}_A, t)\,\vec{j} + w(\vec{r}_A, t)\,\vec{k}$$

The <u>acceleration</u> of A is given by:

$$\vec{a}_A = \frac{d\vec{V}_A}{dt} = \frac{\partial \vec{V}_A}{\partial t} + \frac{\partial \vec{V}_A}{\partial x}\underbrace{\frac{dx_A}{dt}}_{u} + \frac{\partial \vec{V}_A}{\partial y}\underbrace{\frac{dy}{dt}}_{v} + \frac{\partial \vec{V}_A}{\partial z}\underbrace{\frac{dz}{dt}}_{w}$$

$$\vec{a}_A = \frac{\partial \vec{V}_A}{\partial t} + u\frac{\partial \vec{V}_A}{\partial x} + v\frac{\partial \vec{V}_A}{\partial y} + w\frac{\partial \vec{V}_A}{\partial z}$$

In general:

$$\vec{a} = \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}} + \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}}$$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

In shorthand notation:

$$\vec{a} = \frac{D\vec{V}}{Dt}$$

where $\frac{D}{Dt} = \frac{\partial ()}{\partial t} + u\frac{\partial ()}{\partial x} + v\frac{\partial ()}{\partial y} + w\frac{\partial ()}{\partial z}$

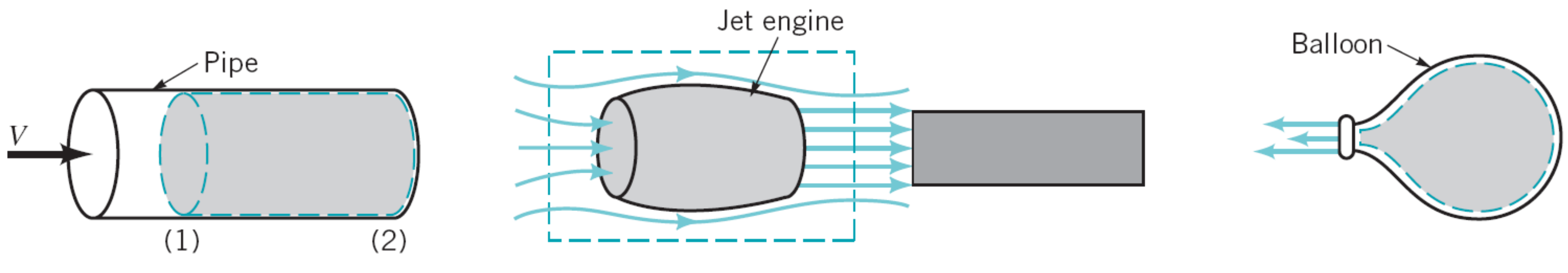
is <u>the material derivative</u>

$$\frac{D}{Dt} = \frac{\partial ()}{\partial t} + (\vec{V}\cdot\nabla)(\ )$$

The gradient operator $\nabla() = \frac{\partial ()}{\partial x}\vec{i} + \frac{\partial ()}{\partial y}\vec{j} + \frac{\partial ()}{\partial z}\vec{k}$

is a <u>vector</u> <u>operator</u>

# Control volume and system




(a) (b) (c)

Control volume surface  System at time $t_1$  System at time $t_2 > t_1$

System: A collection of matter of fixed identity (specified mass)

Control volume A specified volume in space through which fluid may flow

# Reynolds Transport Theorem (RTT)

Most physical laws are written for <u>systems</u>

RTT: relates <u>System</u> to $\underbrace{\text{Control Volume}}_{\text{preferred in fluid mechanics}}$

$$B = m \cdot b$$

extensive property (i.e., mass, momentum, temperature, velocity, energy, etc.)

intensive property

Examples:

$$B = m \rightsquigarrow b = 1$$

$$B = mV \rightsquigarrow b = V$$

$$B = \frac{1}{2} m V^2 \rightsquigarrow b = \frac{V^2}{2}$$

# Reynolds Transport Theorem (RTT)

$$B_{sys} = \int_{sys} b \, dm = \underbrace{\int_{sys} b \cdot \rho \cdot dV}_{\text{sum of quantity } B \text{ contained in all small fluid elements within the system}}$$

Time rate of change of B within the fluid system:

$$\frac{dB_{sys}}{dt} = \frac{d\left(\int_{sys} \rho \, b \, dV\right)}{dt}$$

Relation?

Time rate of change of B within the control volume

$$\frac{dB_{cv}}{dt} = \frac{d\left(\int_{cv} \rho \, b \, dV\right)}{dt}$$

# Example

(a)

Q: Fluid flows from a fire extinguisher tank. The system consists of all fluid in the tank, the control volume is defined by the outer surface of the tank. Consider the extensive property mass (B=m, b=1). How do the time rate of B in the system relate to the time rate of B in the control volume ?

$$\frac{dB_{sys}}{dt} = \frac{d\left(\int_{sys} \rho \, d\forall\right)}{dt}$$

$$\rightarrow \frac{dB_{sys}}{dt} = 0$$

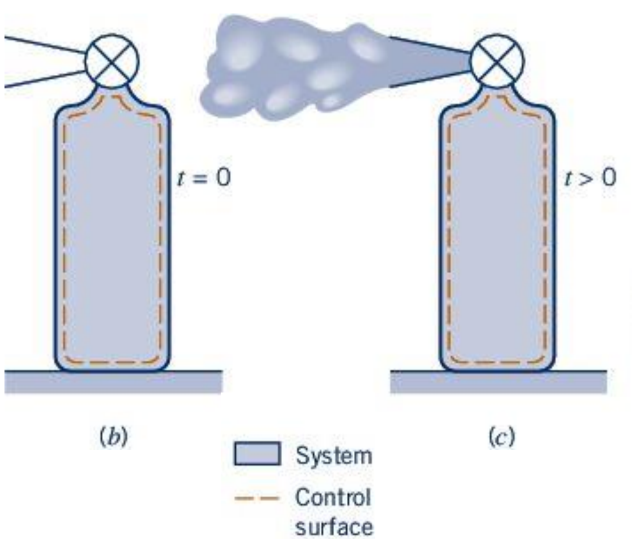
Conservation of mass!

But, at the same time:

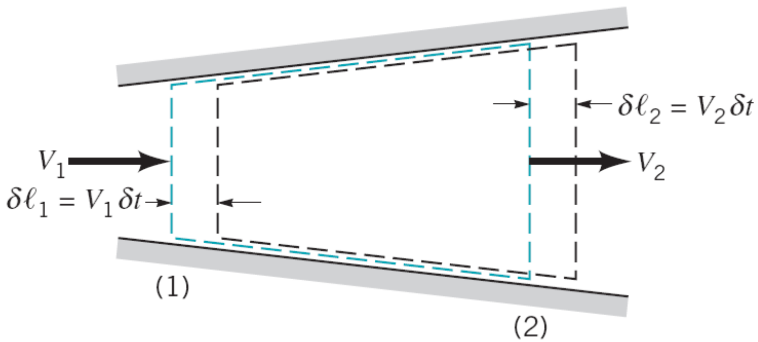
$$\frac{dB_{cv}}{dt} = \frac{d\left(\int_{cv} \rho \, d\forall\right)}{dt}$$

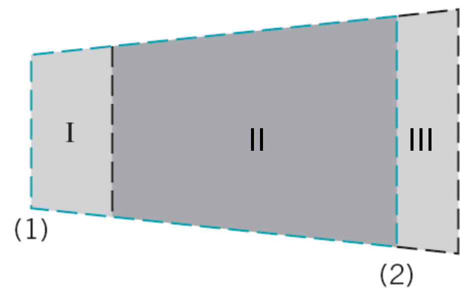
$$\rightarrow \frac{dB_{cv}}{dt} < 0$$

Even if the C.V. at a specific moment in time coincides with the system, the rate of change of B within the C.V. is not necessarily that of the system



# Derivation of the Reynolds Transport Theorem (RTT)




Fixed control surface and system boundary at time $t$

System boundary at time $t + \delta t$

@ time $= t$

Sys = C.V. = I + II

@ time $= t + \delta t$

Sys = II + III

C.V. = I + II

Objective: evaluate $\dfrac{dB_{sys}}{dt}$

relate to $\dfrac{dB_{cv}}{dt}$

$$\frac{\delta B_{sys}}{\delta t} = \frac{B_{sys}(t+\delta t) - B_{sys}(t)}{\delta t} \qquad (1)$$

$$B_{sys}(t+\delta t) = B_{II}(t+\delta t) + B_{III}(t+\delta t)$$

$$= \underbrace{B_{I}(t+\delta t) + B_{II}(t+\delta t)}_{B_{cv}(t+\delta t)} + B_{III}(t+\delta t) - B_{I}(t+\delta t) \qquad (2)$$

$$B_{sys}(t) = B_{cv}(t) \qquad (3)$$

From equations (1), (2) and (3):

$$\Rightarrow \frac{\delta B_{sys}}{\delta t} = \frac{B_{cv}(t+\delta t) + B_{III}(t+\delta t) - B_{I}(t+\delta t) - B_{cv}(t)}{\delta t}$$

$$= \frac{B_{cv}(t+\delta t) - B_{cv}(t)}{\delta t} + \frac{B_{III}(t+\delta t)}{\delta t} - \frac{B_{I}(t+\delta t)}{\delta t}$$

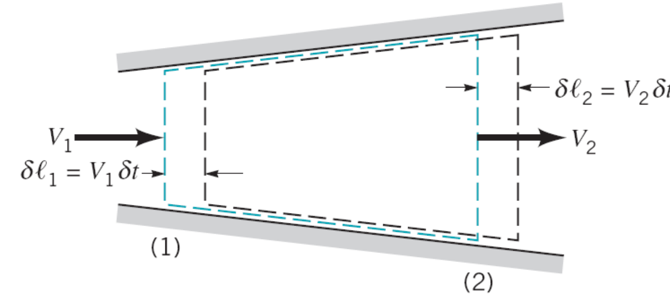
$$\lim_{\delta t \to 0} \frac{\delta B_{sys}}{\delta t} = \frac{D B_{sys}}{D t}$$

material derivative

$$\lim_{\delta t \to 0} \frac{B_{cv}(t+\delta t) - B_{cv}(t)}{\delta t} = \frac{\partial B_{cv}}{\partial t}$$

Also, $B_{III}(t+\delta t) = \rho_2 b_2 \delta V_{III} = \rho_2 b_2 A_2 V_2 \delta t$

and $B_I(t+\delta t) = \rho_1 b_1 \delta V_I = \rho_1 b_1 A_1 V_1 \delta t$




Rate of outflow: $\dot{B}_{out} = \lim\limits_{\delta t \to 0} \dfrac{B_{III}(t+\delta t)}{\delta t} = \rho_2 b_2 A_2 V_2$

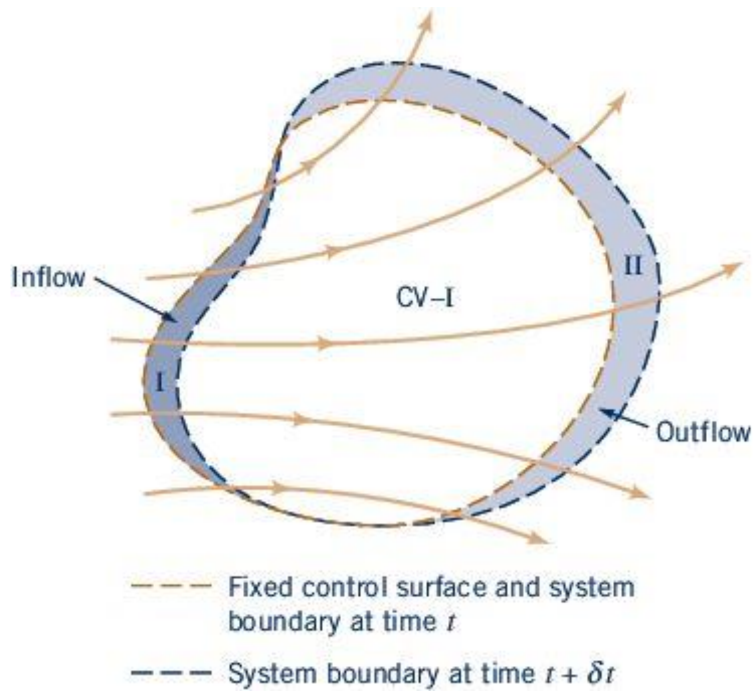
Rate of inflow: $\dot{B}_{in} = \lim\limits_{\delta t \to 0} \dfrac{B_I(t+\delta t)}{\delta t} = \rho_1 b_1 A_1 V_1$

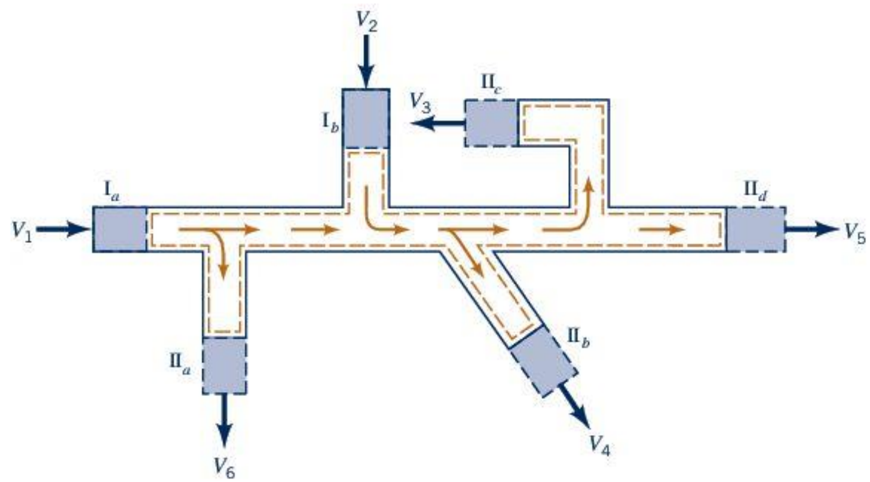
Hence, $$\frac{DB_{sys}}{Dt} = \frac{\partial B_{cv}}{\partial t} + \dot{B}_{out} - \dot{B}_{in}$$

or $$\frac{DB_{sys}}{Dt} = \underbrace{\frac{\partial B_{cv}}{\partial t}}_{\text{change of B within the C.V.}} + \underbrace{\rho_2 b_2 A_2 V_2}_{\text{rate of flow of B out of the C.V.}} - \underbrace{\rho_1 b_1 A_1 V_1}_{\text{rate of flow of B into the C.V.}}$$

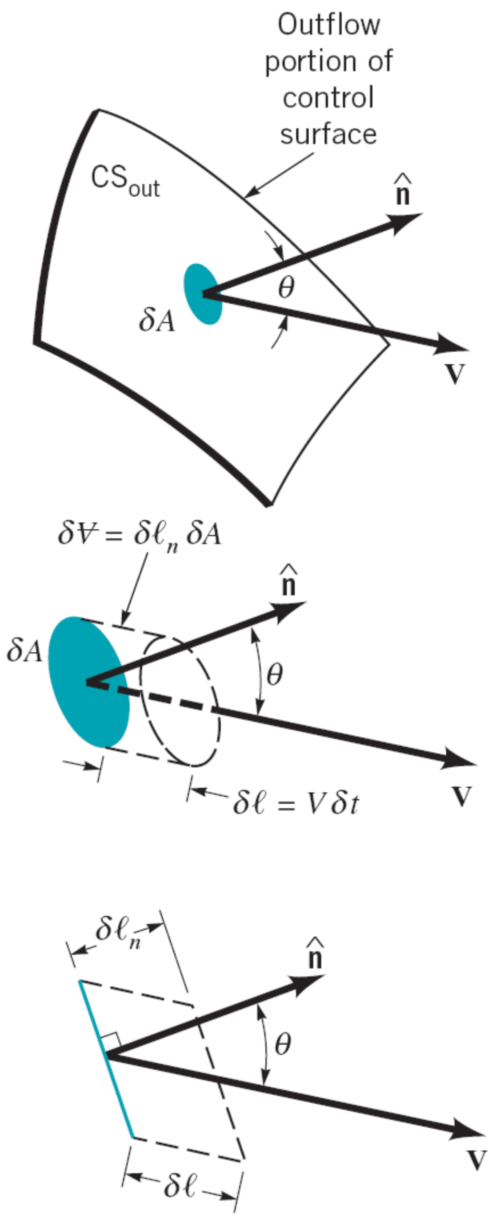
Reynolds Transport Theorem for 1-D flow

# But what happens in control volumes with complex geometries or with multiple in- and outlets?



# Outflow across a typical portion of the control surface




$$\delta B_{out} = b \cdot \rho \cdot \delta \forall_{out}$$

$$\delta \forall_{out} = \delta A \cdot \delta \ell_n = \delta A \cdot \delta \ell \cdot \cos\theta$$

$$= \delta A \cdot V \cdot \delta t \cdot \cos\theta$$

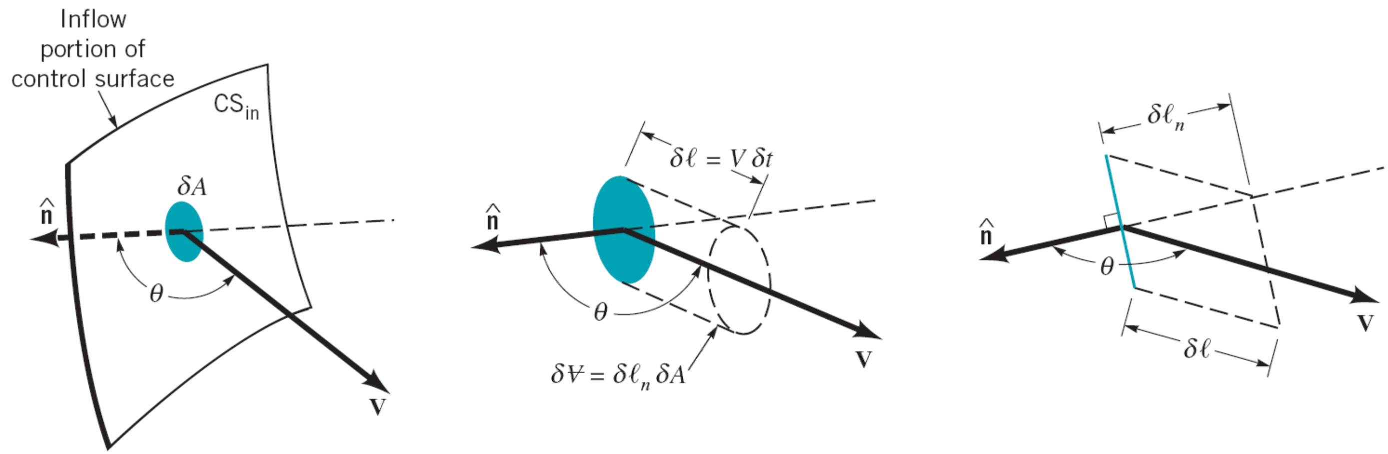
$$= \delta A \cdot \vec{V} \cdot \vec{n} \cdot \delta t$$

$$\therefore \quad \delta B_{out} = b \cdot \rho \cdot \vec{V} \cdot \vec{n} \cdot \delta A \cdot \delta t$$

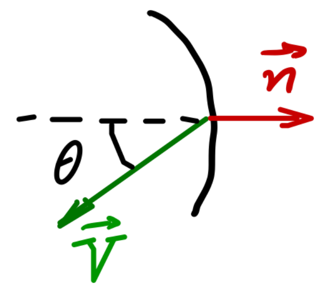
$$\Rightarrow \quad \frac{\delta B_{out}}{\delta t} = b \cdot \rho \cdot \vec{V} \cdot \vec{n} \, \delta A$$

$$\Rightarrow \quad \dot{B}_{out} = \int_{c.s.} \rho \cdot b \cdot \vec{V} \cdot \vec{n} \, dA$$

# Inflow across a typical portion of the control surface



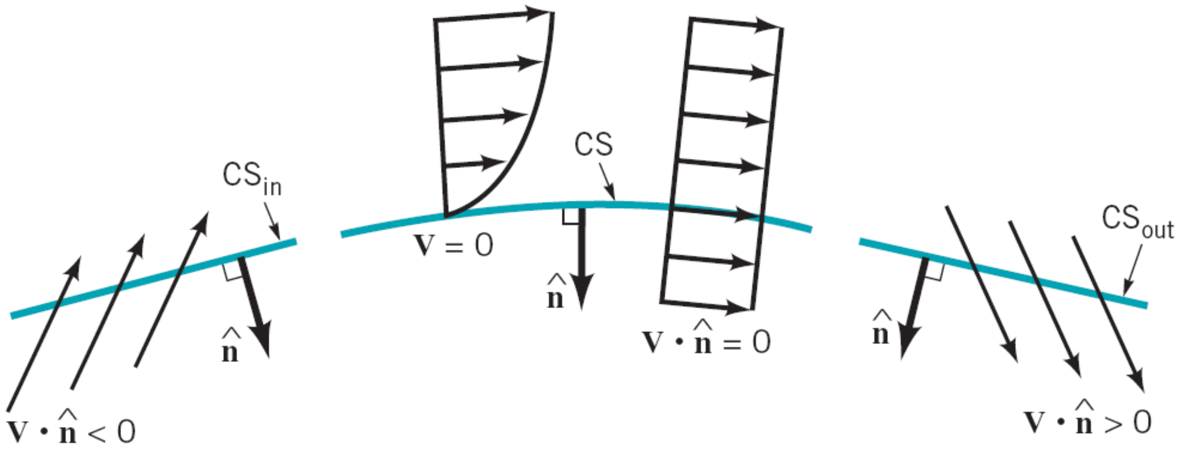

For inflow, $\theta > 90°$




$$\delta \forall_{in} = \delta A \cdot \delta \ell_n = \delta A \cdot \delta \ell \cdot \cos\theta$$
$$= \delta A \cdot V \cdot \delta t \, \cos\theta$$
$$= - \, \delta A \cdot \vec{V} \cdot \vec{n} \cdot \delta t$$

Hence $$\dot{B}_{in} = -\int_{C.S.} \rho \cdot b \cdot \vec{V} \cdot \vec{n} \, dA$$

# Reynolds Transport Theorem for the general 3D case

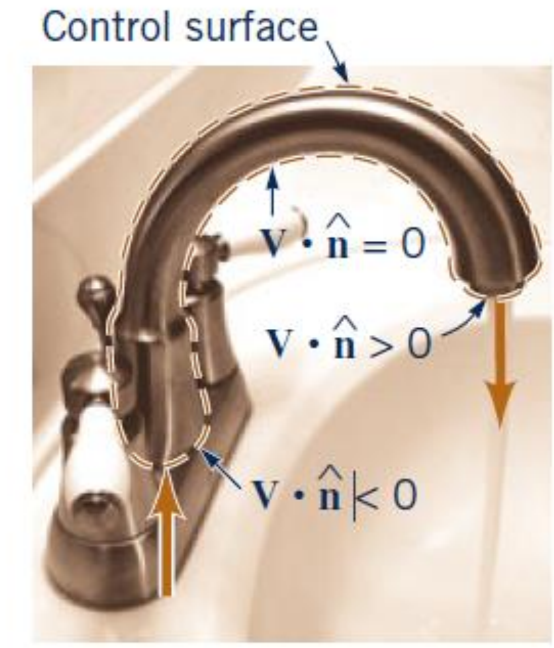



$$\frac{DB_{sys}}{Dt} = \frac{\partial B_{cv}}{\partial t} + \dot{B}_{out} - \dot{B}_{in}$$

or $$\frac{DB_{sys}}{Dt} = \frac{\partial B_{cv}}{\partial t} + \int_{\substack{c.s.\\ outflow}} \rho \cdot b \cdot \vec{V} \cdot \vec{n}\, dA - \left( - \int_{\substack{c.s.\\ inflow}} \rho b \vec{V} \cdot \vec{n}\, dA \right)$$

$$\therefore \quad \frac{DB_{sys}}{Dt} = \frac{\partial B_{cv}}{\partial t} + \int_{c.s.} \rho b\, \vec{V} \cdot \vec{n}\, dA$$

# Reynolds Transport Theorem for the general 3D case




$$\frac{DB_{sys}}{Dt} = \frac{\partial B_{cv}}{\partial t} + \int_{C.S.} \rho b \, \vec{V} \cdot \vec{n} \, dA$$